Chapter 2

An Introduction to

Partial Differential Equations (PDE)

First Session Contents:

1) Definitions and principles
2) Cauchy Problem
3) Second Order PDEs
4) Cauchy-Kowalewsky Theorem

1

Definitions and principles

Partial Differential Equation (PDE)

$$f(x,\ y,\ \ldots,\ u,\ u_x,\ u_y,\ \ldots,\ u_{xx},\ u_{yy},\ \ldots) = 0$$

$$\hookrightarrow \quad u = u(x, y, \ldots)$$

$$uu_{xy} + u_x = y$$

$$u_{xx} + 2yu_{xy} + 3xu_{yy} = 4\sin x$$

$$(u_x)^2 + (u_y)^2 = 1$$

$$u_{xx} - u_{yy} = 0 \longrightarrow \begin{array}{l} u(x,y) = (x+y)^3 \\ u(x,y) = \sin(x-y) \end{array}$$

2

Definitions and principles

Partial Differential Equation (PDE)

$$f(x,\ y,\ \ldots,\ u,\ u_x,\ u_y,\ \ldots,\ u_{xx},\ u_{yy},\ \ldots) = 0$$

$$\hookrightarrow \quad u = u(x, y, \ldots)$$

Order of PDE :

higher order of partial derivative

$$u_{xx} + 2xu_{xy} + u_{yy} = e^y \longrightarrow 2$$

$$u_{xxy} + xu_{yy} + 8u = 7y \longrightarrow 3$$

3

Definitions and principles

General form of PDE

$$Au_{xx} + 2Bu_{xy} + cu_{yy} = \Phi = Du_x + Eu_y + Fu + G$$

A, B, C and ϕ are the finction of x, y, u, u_x and u_y

Definition

1- Linear Equation
2- Quasi Linear Equation
3- Non Linear Equation
4- Homogeneous and Non Homogeneous Equations

4

Definitions and principles

General form of PDE

$$Au_{xx} + 2Bu_{xy} + cu_{yy} = \Phi = Du_x + Eu_y + Fu + G$$

A, B, C and ϕ are function of x, y, u, u_x and u_y

Linear PDE
If it is linear in the unknown functions and time derivatives with coefficients depending on x & y

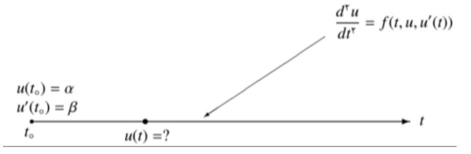
Quasi-Linear (PDE of order m)
If it is linear in the derivatives of order m with coefficients depending on x & y and the derivatives of order less than m

Homogenous
If G=0 the PDE is Homogenous otherwise is called Non- Homogenous

Cauchy problem

Second order ODE $\longrightarrow \dfrac{d^2u}{dt^2} = f(t, u, \dfrac{du}{dt})$

Initial Conditions $\longrightarrow u(t_0) = \alpha, \quad \dfrac{du}{dt}(t_0) = \beta$



Question

Can we determine u at any time (off of t_0) with these initial values?

Cauchy problem

Taylor serious expansion for solving the equation based on an initial points

$$u(t) = u(t_0) + \left.\frac{du}{dt}\right|_{t_0}(t - t_0) + \frac{1}{2!}\left.\frac{d^2u}{dt^2}\right|_{t_0}(t - t_0)^2 + \frac{1}{3!}\left.\frac{d^3u}{dt^3}\right|_{t_0}(t - t_0)^3 + \cdots$$

Need to calculate the higher order derivatives

$$\begin{cases} \left.\dfrac{d^2u}{dt^2}\right|_{t_0} = f(t_0, u(t_0), u'(t_0)) \\ \left.\dfrac{d^3u}{dt^3}\right|_{t_0} \\ \dots \end{cases}$$

Cauchy problem

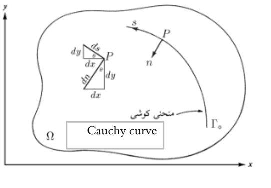
$$Au_{xx} + Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

A, B and C are functions of x, y, u, u_x and u_y

Γ_0 is a curve in x, y plate

Initial values on Γ_0 curve

$\left.\begin{matrix} u \\ \dfrac{\partial u}{\partial n} \end{matrix}\right\}$ are known



Question

Can we determine $u(x, y)$ based on the initial condition on Γ_0 curve?

Cauchy problem

$$Au_{xx} + Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

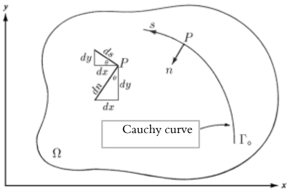
A, B and C are functions of x, y, u, u_x and u_y

Γ_0 is a curve in x, y plate

$$\begin{cases} x_0 = x_0(s) \\ y_0 = y_0(s) \end{cases}$$

Initial values on Γ_0 curve

$$\begin{cases} u = f(s) \\ \dfrac{\partial u}{\partial n} = g(s) \end{cases}$$



9

Cauchy problem

Taylor series expansion of $u(x, y)$ about the point P

$$u(x,y) = u(P) + (x - x_0)u_x(P) + (y - y_0)u_y(P) + \frac{1}{2!}\Big[(x - x_0)^2 u_{xx}(P) + (x - x_0)(y - y_0)u_{xy}(P) + (y - y_0)^2 u_{yy}(P)\Big] + \dots$$

If we could determine the u_x, u_y, u_{xx}, u_{xy} and u_{yy},

then the $u(x, y)$ can be calculated as a surface in (x, y, u) *space*

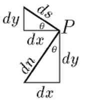
Along the Γ_0 curve:

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = \frac{df}{ds} \quad \& \quad \frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\frac{dx}{dn} + \frac{\partial u}{\partial y}\frac{dy}{dn} = g$$

10

Cauchy problem

$$\left.\begin{aligned} \frac{dx}{ds} = -\cos\theta \quad &\& \quad \frac{dy}{ds} = \sin\theta \\ \frac{dy}{dn} = -\cos\theta \quad &\& \quad \frac{dx}{dn} = -\sin\theta \end{aligned}\right\} \quad \begin{aligned} \frac{dx}{dn} &= -\frac{dy}{ds} \\ \frac{dy}{dn} &= \frac{dx}{ds} \end{aligned}$$



$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\frac{dx}{dn} + \frac{\partial u}{\partial y}\frac{dy}{dn} = g \longrightarrow \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}\frac{dy}{ds} + \frac{\partial u}{\partial y}\frac{dx}{ds} = g$$

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = \frac{df}{ds} \longrightarrow \frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = \frac{df}{ds}$$

$$\begin{bmatrix} -\dfrac{dy}{ds} & \dfrac{dx}{ds} \\ \dfrac{dx}{ds} & \dfrac{dy}{ds} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} g \\ \dfrac{df}{ds} \end{bmatrix} \Longrightarrow$$

This equation will have an answer if $\Delta \neq 0$

11

Cauchy problem

by solving this equation, u_x and u_y can be determined along the Γ_0 curve

Higher order differentiation
$$\begin{cases} \dfrac{d}{ds}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial x^2}\dfrac{dx}{ds} + \dfrac{\partial^2 u}{\partial x \partial y}\dfrac{dy}{ds} \\ \dfrac{d}{ds}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial^2 u}{\partial x \partial y}\dfrac{dx}{ds} + \dfrac{\partial^2 u}{\partial y^2}\dfrac{dy}{ds} \end{cases}$$

The based equation $\longrightarrow$ $A\dfrac{\partial^2 u}{\partial x^2} + B\dfrac{\partial^2 u}{\partial x \partial y} + C\dfrac{\partial^2 u}{\partial y^2} = F(x, y, u, u_x, u_y)$

$$\begin{vmatrix} \dfrac{dx}{ds} & \dfrac{dy}{ds} & 0 \\ 0 & \dfrac{dx}{ds} & \dfrac{dy}{ds} \\ A & B & C \end{vmatrix} = C\left(\frac{dx}{ds}\right)^2 - B\frac{dx}{ds}\frac{dy}{ds} + A\left(\frac{dy}{ds}\right)^2$$

This equation will have an answer if $\Delta \neq 0$

12

Cauchy problem

Characteristic equation $\Rightarrow$ $A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0$

Necessary condition for second order derivative

$\rightarrow$ $\Gamma_0 \neq$ the characteristic *curve*

Final solution

$$u(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\frac{\partial^n u_0}{\partial x_0^k \partial y_0^{n-k}}(x-x_0)^k(y-y_0)^{n-k}$$

Second order PDEs types

Three main PDE equations in fluid mechanics and heat transfer

$\phi_{xx} + \phi_{yy} = 0$ $\longrightarrow$ Potential flow

$u_{tt} = c^2 u_{xx}$ $\longrightarrow$ Wave equation

$u_t = u_{xx}$ $\longrightarrow$ Unsteady heat transfer equation

- All of these equations are linear and second order, but they show different physical behavior
- The number and types of IC and BC are different for each type

Classification of PDE's

$Au_{xx} + Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$

A, B and C are functions of x, y, u, u_x and u_y

Characteristic Equation: $A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0$

$\Delta = B^2 - 4AC$

$\Delta > 0$ $\rightarrow$ Two characteristic equations called **Hyperbolic**

$\Delta = 0$ $\rightarrow$ One characteristic equation called **Parabolic**

$\Delta < 0$ $\rightarrow$ No characteristic equation called **Elliptic**

Canonical Forms of PDE's

coordinate transformation for uncoupling PDEs

$(x, y) \rightarrow (\xi, \eta)$

$\left.\begin{array}{l}\xi = cte \\ \eta = cte\end{array}\right\}$ characteristic equation in transformed coordinate

In the main PDE.

$$B^2 - 4AC \begin{cases} \Delta > 0 & \rightarrow \text{Two characteristic equations} \\ \Delta = 0 & \rightarrow \text{One characteristic equation} \\ \Delta < 0 & \rightarrow \text{No characteristic equation} \end{cases}$$

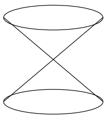
Canonical Forms of PDEs

Transformed equations in new coordinate system is defined as Canonical Form

$$\begin{cases} \Delta > 0 \rightarrow \text{Hyperbolic} \rightarrow & u_{\xi\eta} = H(\xi, \eta, u, u_\xi, u_\eta) \\ \Delta = 0 \rightarrow \text{Parabolic} \rightarrow & u_{\xi\xi} \text{ or } u_{\eta\eta} = H(\xi, \eta, u, u_\xi, u_\eta) \\ \Delta < 0 \rightarrow \text{Elliptic} \rightarrow & u_{\xi\xi} + u_{\eta\eta} = H(\xi, \eta, u, u_\xi, u_\eta) \end{cases}$$

Name of each type is based on this equation

$$Ax^{۲} + Bxy + Cy^{۲} + Dx + Ey + F = ۰$$



17

Example

$$\nabla^{۲} u = u_{xx} + u_{yy} = ۰ \quad \begin{cases} A = ۱ \\ B = ۰ \\ C = ۱ \end{cases} \quad B^{۲} - ۴AC = -۴ < ۰ \quad \frac{dy}{dx} = \pm i$$

$$u_{xx} = u_t \quad \begin{cases} A = ۱ \\ B = C = ۰ \end{cases} \quad B^{۲} - ۴AC = ۰ \quad \frac{dt}{dx} = ۰$$

$$u_{xx} - u_{tt} = ۰ \quad \begin{cases} A = ۱ \\ B = ۰ \\ C = -۱ \end{cases} \quad B^{۲} - ۴AC = ۴ > ۰ \quad \frac{dt}{dx} = \pm ۱$$

18

General classification of second order PDEs

$$\sum_{i,j=۱}^{n} A_{ij} u_{x_i x_j} + \sum_{i=۱}^{n} B_i u_{x_i} + Fu = G$$

$$u_{x_i x_j} = u_{x_j x_i} \longrightarrow A = [A_{ij}]_{n\times n} : \text{Symmetric matrix}$$

$$\text{Eigenvalues} \longrightarrow |A - \lambda I| = ۰$$

z = Number of (eigenvalues = 0)
p = Number of (eigenvalues > 0)

$$\begin{cases} z = 0 \,\&\, p = 1 \text{ or } z = 0 \,\&\, p = n-1 & \rightarrow \text{Hyperbolic} \\ z > 0 \,(|A| = 0) & \rightarrow \text{Parabolic} \\ z = 0 \,\&\, p = 0 \text{ or } z = 0 \,\&\, p = n & \rightarrow \text{Elliptic} \end{cases}$$

19

Example

$$۳u_{x_۱x_۱} + ۸u_{x_۲x_۲} + ۶u_{x_۲x_۳} = ۰$$

$$۳u_{x_۱x_۱} + ۸u_{x_۲x_۲} + ۳u_{x_۲x_۳} + ۳u_{x_۳x_۲} = ۰$$

$$A = \begin{bmatrix} ۳ & ۰ & ۰ \\ ۰ & ۸ & ۳ \\ ۰ & ۳ & ۰ \end{bmatrix} \quad |A - \lambda I| = (۳ - \lambda)(-\lambda(۸ - \lambda) - ۹) = ۰$$

$$\lambda = ۳, ۹, -۱$$

Hyperbolic Equation

20

Domain of dependence and range of influence

$$\left.\begin{aligned} A f_{xx} + B f_{xy} + C f_{yy} &= 0 \\ d(f_x) &= f_{xx}dx + f_{xy}dy \\ d(f_y) &= f_{xy}dx + f_{yy}dy \end{aligned}\right\} \quad f_{xy} = \frac{\begin{vmatrix} A & 0 & C \\ dx & df_x & 0 \\ 0 & df_y & dy \end{vmatrix}}{\begin{vmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}}$$

$$\begin{vmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0 \longrightarrow \frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \quad \text{Characteristic equation}$$

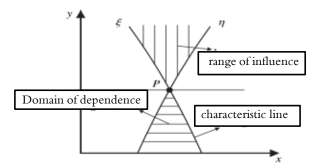
$$\begin{vmatrix} A & 0 & C \\ dx & df_x & 0 \\ 0 & df_y & dy \end{vmatrix} = 0 \longrightarrow \frac{df_y}{df_x} = \frac{-A}{C}\left.\frac{dy}{dx}\right)_{char}$$

Domain of dependence and range of influence

Hyperbolic equation
$$\begin{cases} \dfrac{df_y}{df_x} = \dfrac{-A}{C}\,\dfrac{B + \sqrt{B^2 - 4AC}}{2A} \longrightarrow \text{Along the first characteristic line} \\ \dfrac{df_y}{df_x} = \dfrac{-A}{C}\,\dfrac{B - \sqrt{B^2 - 4AC}}{2A} \longrightarrow \text{Along the second characteristic line} \end{cases}$$

PDEs $\xrightarrow{\text{Along the characteristic lines}}$ **ODEs** (compatibility equations)

For elliptic equations, there is no characteristic line. Thereafter all the domain affect point P and depend on it.



Example

$$u_{xx} - 3u_{xy} - 4u_{yy} = 0 \qquad u(x,0) = x \qquad u_y(x,0) = 2$$

$$\text{Slope of characteristics lines} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-3 \pm 5}{2} = \begin{cases} 1 & C^+ \\ -4 & C^- \end{cases}$$

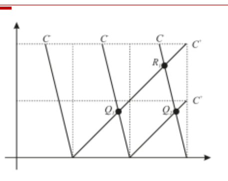
$$C^+: \quad \frac{du_y}{du_x} = \frac{-1}{-4}(1) = \frac{1}{4} \Longrightarrow u_y = \frac{1}{4}u_x + C_1$$

$$C^-: \quad \frac{du_y}{du_x} = \frac{-1}{-4}(-4) = -1 \Longrightarrow u_y = -u_x + C_2$$

Example

$$u_y = \frac{1}{4}u_x + C_1$$

$$u_y = -u_x + C_2$$



$$(1,0) \qquad 2 = \frac{1}{4} \times 1 + C_1 \Longrightarrow C_1 = \frac{7}{4}$$

$$(2,0) \qquad 2 = -1 + C_2 \Longrightarrow C_2 = 3$$

$$u_x = 1 \qquad u_y = 2 \qquad u = x + 2y$$

Cauchy Kowalewsky theorem

Partial differential equation:

$$u_{yy} = F(y, x_1, x_2, x_3, \ldots, x_n, u, u_y, u_{x_1}, u_{x_2}, \ldots, u_{x_n}$$
$$, u_{x_1 y}, u_{x_2 y}, \ldots, u_{x_n y}, u_{x_1 x_1}, u_{x_2 x_2}, \ldots, u_{x_n x_n})$$

Initial conditions: $\begin{cases} u = f(x_1, x_2, \ldots, x_n) \\ u_y = g(x_1, x_2, \ldots, x_n) \end{cases}$

A theorem stating that the Cauchy problem has a (unique) analytic solution locally if the functions occurring in the differential equation (or system of differential equations) and all the initial data, together with the non-characteristic initial surface, are analytic.

25

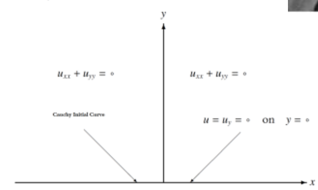
Adamar's Counterexample for Cauchy Problem

$$u_{xx} + u_{yy} = 0$$

$$u(x, 0) = 0$$

$$u_y(x, 0) = n^{-1} \sin nx$$

$$u(x, y) = n^{-2} \sinh ny \sin nx$$



26